

MAS317 Linear Algebra II, 2007

Solutions to Coursework 4

1. (i) By cor. to (2.18) in notes, I_n can be obtained from A by a sequence of elementary row operations. Hence A can be obtained from I_n using the inverses of elementary these row operations.

But the inverse of an elementary row operation is an elementary row operation so A can be obtained from I_n by a sequence of elementary row operations. [5 marks]

(ii) Let $A = \rho_k \dots \rho_1(I_n)$; then $AB = \rho_k \dots \rho_1(I_n)B = \rho_k \dots \rho_1(B)$.

So AB and B are row equivalent and so they have the same row space and the row rank of AB is equal to the row rank of B .

[5 marks]

(iii) Now A^T is also invertible (its inverse is $(A^{-1})^T$), and $(CA)^T = A^T C^T$.

Applying (ii) to $A^T C^T$, the row rank of $A^T C^T$ is equal to the row rank of C^T ;

that is, the row rank of $(CA)^T$ is equal to the row rank of C^T .

Now row rank = column rank, and so the column rank of $(CA)^T$ is equal to the column rank of C^T .

Also column rank of transpose = row rank of matrix, and so the row rank of CA is equal to the row rank of C . [10 marks]

2. (i) $x(1, 0, 1) + y(1, 3, 0) + z(2, 0, 1) = (a, b, c)$ iff

$$\begin{array}{rclclcl} x + y + 2z & = & a & & y + z & = & a - c & & z = a - b/3 - c \\ 3y + 0 & = & b & \text{iff} & y & = & 2 & \text{iff} & y = b/3 \\ x + 0 + z & = & c & & y + z & = & 5 & & x = -a + b/3 + 2c \end{array}$$

So the coordinate vector for (a, b, c) w.r.t. $\mathcal{B}$ is

$(-a + b/3 + 2c, b/3, a - b/3 - c)^T$. [10 marks]

(ii) The change of basis matrix from $\mathcal{C}$ to $\mathcal{B}$ has columns equal to the coordinate vector of each of the vectors in $\mathcal{C}$ in order, and so, from (i), the change of basis matrix from $\mathcal{C}$ to $\mathcal{B}$ is [20 marks]

$$\begin{pmatrix} 4/3 & -1/3 & 1 \\ 1/3 & 2/3 & 0 \\ -1/3 & 1/3 & 0 \end{pmatrix}.$$

3. $f((x_1, x_2, x_3), (y_1, y_2, y_3)) = 2x_1y_2 - 2x_2y_1 - x_2y_3 + x_3y_2.$

(i) $(f; E) = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$ [5 marks]

(ii) $(f; \mathcal{B}) = (M_{\mathcal{B}}^E)^T(f; \mathcal{E})M_{\mathcal{B}}^E$ [10 marks]
 $= \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 6 & 3 \\ -6 & 0 & -4 \\ -3 & 4 & 0 \end{pmatrix}.$

Alternatively calculate $(f; \mathcal{B})$ from the definition:

The ij entry is $f(v_i, v_j)$ for all i, j where $\mathcal{B} = v_1, v_2, v_3$.

(iii) It is not possible to find a basis $\mathcal{C}$ for $\mathbb{R}^3$ such that $(f; \mathcal{C})$ has two rows of zeros as $(f; \mathcal{E})$ has row rank greater than 1, and hence $(f; \mathcal{C})$ has row rank greater than 1 for all $\mathcal{C}$ by (3.12). [5 marks]

(iv) $f((x_1, x_2, x_3), (x_1, x_2, x_3)) = 2x_1x_2 - 2x_2x_1 - x_2x_3 + x_3x_2 = 0$ for all (x_1, x_2, x_3) . As the diagonal entries in $(f; \mathcal{C})$ are all of the form $f(u, u)$, they are zero for every basis $\mathcal{C}$ for $\mathbb{R}^3$. [5 marks]

4. (i) If two of u_1, u_2, u_3 are equal then interchanging them does not change u_1, u_2, u_3 and so $f(u_1, u_2, u_3) = -f(u_1, u_2, u_3)$ and hence $f(u_1, u_2, u_3) = 0$. [5 marks]

(ii) If $u_1 = au_2 + bu_3$ with $a, b \in \mathbb{R}$ then, using (i),

$$f(u_1, u_2, u_3) = f(au_2 + bu_3, u_2, u_3) = af(u_2, u_2, u_3) + bf(u_3, u_2, u_3) = 0.$$
 [5 marks]

$$(1, 1, 2) = (1/2)(2, 2, 0) + (2/3)(0, 0, 3) \text{ and so}$$

$$f((1, 1, 2), (2, 2, 0), (0, 0, 3)) = 0. \quad [5 \text{ marks}]$$

Also $(1, 2, 2) = (1, 1, 2) + (0, 1, 0)$. Hence, as linear in first place,

$$\begin{aligned} f((1, 2, 2), (2, 2, 0), (0, 0, 3)) &= f((1, 1, 2) + (0, 1, 0), (2, 2, 0), (0, 0, 3)) \\ &= 0 + f((0, 1, 0), (2, 2, 0), (0, 0, 3)) = f(e_2, 2e_1 + 2e_2, 3e_3) \\ &= 6f(e_2, e_1, e_3) - 6f(e_2, e_2, e_3) = -6f(e_1, e_2, e_3) - 0 = -36. \end{aligned} \quad [10 \text{ marks}]$$